“Memoire sur les lois du mouvement des fluides”

Claude-Louis Navier

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Claude Navier - Biography

- **Born: 10 Feb 1785 in Dijon, France------ Died: 21 Aug 1836 in Paris.**
- **Education: Ecole Polytechnique in 1802 and Ecole des Ponts et Chaussées (1804)**
- **Professor at Ecole des Ponts et Chaussées (1819) – applied mechanics**
- **Elected to the Académie des Sciences in Paris in 1824**

“Navier believed in an industrialized world in which science and technology would solve most of the problems. “

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Professional profile

- Civil engineering (Emiland Gauthey)
- Bridge construction
- Survey and analysis of river flow
- Engineering, elasticity and fluid mechanics
- Contributions to Fourier series and their application to physical problems.

- Work on modifying Euler's equations to take into account forces between the molecules in the fluid.

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A short history of fluid dynamics

- **3rd B.C. , Archimedes, "On Floating Bodies"**
- **15th Century, L. de Vinci, observations**
- **1687, Newton in " Principia", viscosity force α velocity variation**
- **1738, D. Bernoulli, "Hydrodynamics"**
- **1755, L. Euler, perfect flow, equations of continuity and momentum for frictionless fluids which are compressible or incompressible**

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A short history of fluid dynamics

- **1821, C. Navier : derived Navier-Stokes equations, stress tensor**

- **1829, S. Poisson, for viscous fluids**

- **1845, G. Stokes, rederived Navier's results, formulated the non-slip boundary condition (considering friction but non-solvable mathematically)**

- **1851, G. Stokes solved " a spherical particle moving through viscous liquids neglecting inertial forces"**

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Context

From Euler (perfect flow)

$$\rho \frac{D\mathbf{u}}{Dt} = \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla P,$$

To Navier (Viscous and incompressible flow)

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F_x$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + F_y$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \eta\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + F_z.$$

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Navier's slip condition and general formulation

No-slip boundary condition to fluid flow over a solid surface

Validated by number of macroscopic flows but it remains an assumption based on physical principles.

Navier's proposed boundary condition assumes that the velocity, u_z, at a solid surface is proportional to the shear rate at the surface

shear rate at the surface

$$\eta \frac{\partial u_z}{\partial z} = \beta u_z$$

tangent component of fluid velocity

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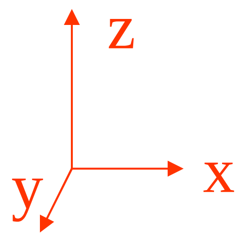
Calculation of the molecular forces developed by the fluid motion

- M is at the interface between wall and fluid (x,y,z)
- M velocity (u,v,w)



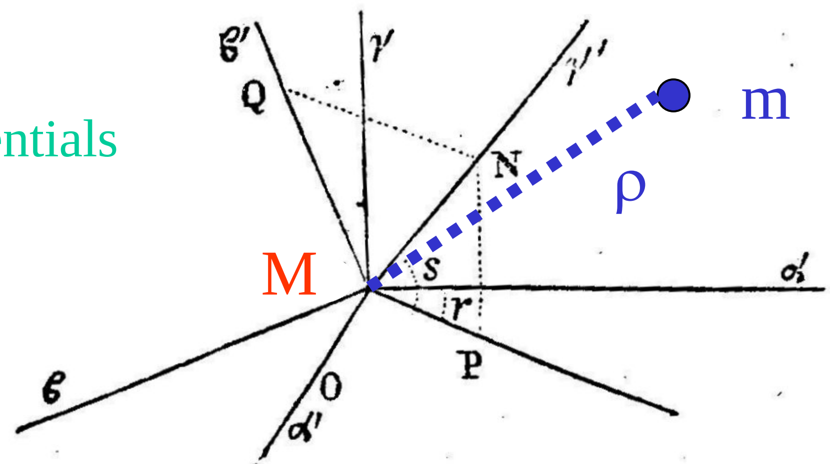
- m is in the fluid:

$$\begin{cases} x+\alpha \\ y+C \\ z+\gamma \end{cases}$$

- m velocity:

$$\begin{cases} u + \frac{du}{dx}\alpha + \frac{du}{dy}\beta + \frac{du}{dz}\gamma, \\ v + \frac{dv}{dx}\alpha + \frac{dv}{dy}\beta + \frac{dv}{dz}\gamma, \\ w + \frac{dw}{dx}\alpha + \frac{dw}{dy}\beta + \frac{dw}{dz}\gamma, \end{cases}$$

Referentials



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Calculation of the molecular forces developed by the fluid motion

Velocity of m moving away from M $\frac{1}{\rho}(\alpha u + \beta v + \gamma w)$

If impulsion to m: Velocity of m $\frac{1}{\rho}(\alpha \delta u + \beta \delta v + \gamma \delta w)$

Σ moment of action between molecule m and M, in Mm direction :

$$\frac{F(\rho)}{\rho^2}(\alpha u + \beta v + \gamma w).(\alpha \delta u + \beta \delta v + \gamma \delta w)$$

Action of the molecule m on M

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Calculation of the molecular forces developed by the fluid motion

1. Integration on the half sphere

2. Referential change

3. Multiply by volume unit

4. Let’s introduce: $$\frac{4.\pi}{6}\int_0^\infty d\rho.\rho^3 F(\rho) = E$$

Where E is a constant given by experiment and depending on the wall and fluid nature.
“E can be expressed as the reciprocal action wall/fluid” translated from Navier.

➔ **Σ moments generated by actions of all molecules m-like to M:**

$$\boxed{E(u\delta u + v\delta v + w\delta w)}$$ 5. x ds^2

6. Integration on the surface of the fluid

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Calculation of the molecular forces developed by the fluid motion

6. General eq : Σ moments applied to incompressible fluid molecules =0

density

acceleration force

$$0=\iiint dx\,dy\,dz\left\{\begin{aligned}&\left[P-\frac{dp}{dx}-\rho\left(\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}\right)\right]\delta u\\&\left[Q-\frac{dp}{dy}-\rho\left(\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}\right)\right]\delta v\\&\left[R-\frac{dp}{dz}-\rho\left(\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dv}{dz}\right)\right]\delta w\end{aligned}\right\}$$

now-called viscosity

$$-\varepsilon\iiint dx\,dy\,dz\left\{\begin{aligned}&3\frac{du}{dx}\frac{\delta du}{dx}+\frac{du}{dy}\frac{\delta du}{dy}+\frac{du}{dz}\frac{\delta du}{dz}+\frac{dv}{dy}\frac{\delta du}{dx}+\frac{dv}{dx}\frac{\delta du}{dy}+\frac{dw}{dz}\frac{\delta du}{dx}+\frac{dw}{dx}\frac{\delta du}{dz}\\&\frac{du}{dx}\frac{\delta dv}{dy}+\frac{du}{dy}\frac{\delta dv}{dx}+\frac{dv}{dx}\frac{\delta dv}{dx}+3\frac{dv}{dy}\frac{\delta dv}{dy}+\frac{dv}{dz}\frac{\delta dv}{dz}+\frac{dw}{dy}\frac{\delta dv}{dz}+\frac{dw}{dz}\frac{\delta dv}{dy}\\&\frac{du}{dx}\frac{\delta dw}{dz}+\frac{du}{dz}\frac{\delta dw}{dx}+\frac{dv}{dy}\frac{\delta dw}{dz}+\frac{dv}{dz}\frac{\delta dw}{dy}+\frac{dw}{dx}\frac{\delta dw}{dx}+\frac{dw}{dy}\frac{\delta dw}{dy}+3\frac{dw}{dz}\frac{\delta dw}{dz}\end{aligned}\right\}$$

(Eq. 1)

$$+\,S\,ds^2.E(u\delta u+v\delta v+w\delta w).$$

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From Eq. 1 => Navier's motion equation

1. Partial integration of eq. 1
2. Continuity equation

$$0 = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$$

"Indefinite equations of motion":

$$P - \frac{dp}{dx} = \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right) - \varepsilon\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right)$$

$$Q - \frac{dp}{dy} = \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right) - \varepsilon\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right)$$

$$R - \frac{dp}{dz} = \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right) - \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right)$$

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From Eq. 1 => Navier's Slip condition

1. Referential change (l,m,n) angle with plan yz, xz and xy

2. Conditions to get δu, δv, δw= 0

$$Eu + \varepsilon\left[\cos.l.2\frac{du}{dx} + \cos.m\left(\frac{du}{dy}+\frac{dv}{dx}\right) + \cos.n\left(\frac{du}{dz}+\frac{dw}{dx}\right)\right] = 0$$

$$Ev + \varepsilon\left[\cos.l\left(\frac{du}{dy}+\frac{dv}{dx}\right) + \cos.m.2\frac{dv}{dy} + \cos.n\left(\frac{dv}{dz}+\frac{dw}{dy}\right)\right] = 0$$

$$Ew + \varepsilon\left[\cos.l\left(\frac{du}{dz}+\frac{dw}{dx}\right) + \cos.m\left(\frac{dv}{dz}+\frac{dw}{dy}\right) + \cos.n.2\frac{dw}{dz}\right] = 0$$

3. No motion of the molecules perpendicular to the wall

$$0 = u.\cos.l + v.\cos.m + w.\cos.n$$

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From Eq. 1 => Navier's Slip condition

$$E u + \varepsilon \left(\cos . l . \frac{du}{dx} + \cos . m . \frac{du}{dy} + \cos . n . \frac{du}{dz} \right) = 0$$

$$E v + \varepsilon \left(\cos . l . \frac{dv}{dx} + \cos . m . \frac{dv}{dy} + \cos . n . \frac{dv}{dz} \right) = 0$$

$$E w + \varepsilon \left(\cos . l . \frac{dw}{dx} + \cos . m . \frac{dw}{dy} + \cos . n . \frac{dw}{dz} \right) = 0$$

Example : If M is perpendicular to z axis

$$E u + \varepsilon \frac{du}{dz} = 0, \quad E v + \varepsilon \frac{dv}{dz} = 0$$

Similar to the common formulation

$$\eta \frac{\partial u_z}{\partial z} = \beta u_z$$

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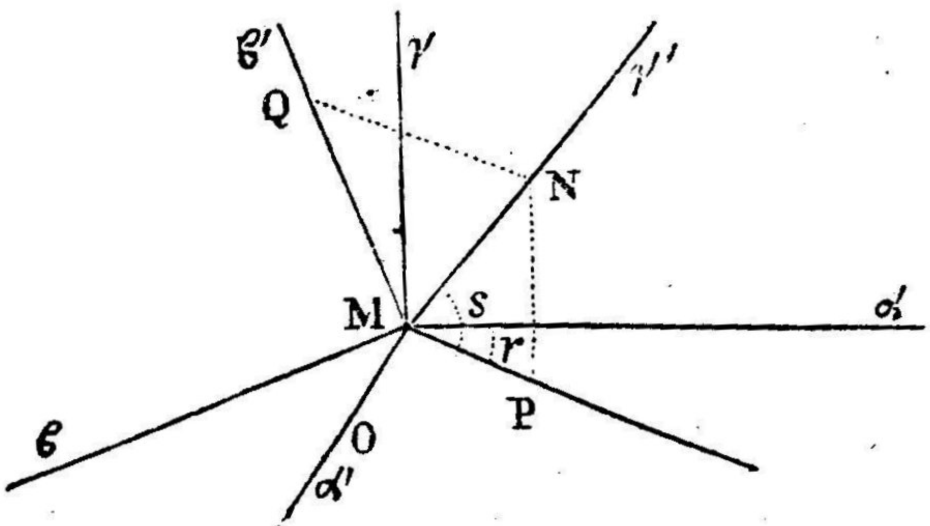
Extra slides for potential explanation

Referential

$$\alpha = \rho \cos.\psi \cos.\varphi \qquad \alpha' = \rho \cos.\psi \cos.\varphi$$
$$\beta = \rho \cos.\psi \sin.\varphi \qquad \beta' = \rho \cos.\psi \sin.\varphi$$
$$\gamma = \rho \sin.\psi\,; \qquad \gamma' = \rho \sin.\psi.$$



Point m definition:

$$u + \frac{du}{dx}\alpha + \frac{du}{dy}\beta + \frac{du}{dz}\gamma,$$
$$v + \frac{dv}{dx}\alpha + \frac{dv}{dy}\beta + \frac{dv}{dz}\gamma,$$
$$w + \frac{dw}{dx}\alpha + \frac{dw}{dy}\beta + \frac{dw}{dz}\gamma,$$

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Extra slides for potential explanation

Velocity of m moving away from M

$$\frac{1}{\rho}(\alpha u + \beta v + \gamma w)$$

Impulsion to m

$$\frac{1}{\rho}(\alpha \delta u + \beta \delta v + \gamma \delta w)$$

Moments of reciprocal actions are:

$$\frac{1}{\rho^2}(\alpha u + \beta v + \gamma w).(\alpha \delta u + \beta \delta v + \gamma \delta w).$$

Sum of moments:

$$\frac{F(\rho)}{\rho^2}(\alpha u + \beta v + \gamma w).(\alpha \delta u + \beta \delta v + \gamma \delta w)$$

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Extra slides for potential explanation

Integration to the half sphere:

$$\frac{4.\mathrm{F}(\rho)}{\rho^2}\left\{\begin{array}{l}
\alpha'^2\left\{\begin{array}{l}(u\sin.^2 r - v\sin.r\cos.r)\,\delta u\\ (-u\sin.r\cos.r + v\cos.^2 r)\,\delta v\end{array}\right\}\\[1em]
\beta'^2\left\{\begin{array}{l}(u\cos.^2 r\sin.^2 s + v\sin.r\cos.r\sin.^2 s + w\cos.r\sin.s\cos.s)\,\delta u\\ (u\sin.r\cos.r\sin.^2 s + v\sin.^2 r\sin.^2 s + w\sin.r\sin.s\cos.s)\,\delta v\\ (u\cos.r\sin.s\cos.s + v\sin.r\sin.s\cos.s + w\cos.^2 s)\,\delta w\end{array}\right\}\\[1em]
\gamma'^2\left\{\begin{array}{l}(u\cos.^2 r\cos.^2 s + v\sin.r\cos.r\cos.^2 s - w\cos.r\sin.s\cos.s)\,\delta u\\ (u\sin.r\cos.r\cos.^2 s + v\sin.^2 r\cos.^2 s - w\sin.r\sin.s\cos.s)\,\delta v\\ (-u\cos.r\sin.s\cos.s - v\sin.r\sin.s\cos.s + w\sin.^2 s)\,\delta w\end{array}\right\}
\end{array}\right\}$$

$$\mathrm{E}(u\,\delta u + v\,\delta v + w\,\delta w)$$

with $\frac{4.\pi}{6}\int_0^\infty d\rho\,.\,\rho^2\,\mathrm{F}(\rho) = \mathrm{E}$

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