A method for the three-dimensional numerical simulation of SuperFluid Helium

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Abstract. Transport phenomena in superfluid helium can be described using the two-fluid *Landau-Khalatnikov* model and the Gorter-Mellink mutual friction. Here we discuss a mathematical formulation of the two-fluid model that uses macroscopic conservation balances of mass, momentum and energy of each species, and assumes local thermodynamic equilibrium. A particularity of this model is that it describes the state of He II as well as that of each of the two-fluid components in terms of pressure p and temperature T, which is convenient for stable numerical solution. The equations of the model form a system of partial differential equations (PDE) that can be written in matrix form for convenience. On this base, a three-dimensional numerical model using a complete and consistent, while still practical, system of PDEs was developed. In the form described, the PDE can be solved using three-dimensional Lagrangian finite element in space supplemented by a Beam-Warming time-²marching algorithm. Once validated, this solver will allow to simulate He II thermal counterflow applied to arbitrary geometry.

1. Introduction

The two-fluid *Landau-Khalatnikov* model has been proven in the past to provide a practical description of transport phenomena in superfluid helium [1]. This is a convenient representation of two co-existing states: a superfluid state that corresponds to the helium atoms that have undergone Bose-Einstein condensation and are in their ground energy state, and a normal fluid state that corresponds to the fraction of helium atoms that are in excited state (phonons and rotons). The equations that describe the two-fluid model are conservation balances (mass, momentum, energy) for each of the two-fluid species. The two fluids interact through different mechanisms depending on the regime of transport. For most technical applications, the interaction has been described by simple expressions such as a power law dependence on the relative velocity as found by Gorter and Mellink [2]. We take here a macroscopic approach to the conservation equations, assuming local thermodynamic equilibrium, so that the state of He II as well as that of each of the two-fluid components can be described by two independent state variables. The interaction between the two fluids is described by the Gorter-Mellink mechanism. As discussed in [3], the conservation balances

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can be cast in the form of a system of partial differential equation (PDE). For reasons that will become clear later, we have chosen to use pressure and temperature as thermodynamic state variables, while the motion is described naturally by the velocity of the normal component, v_n , and the velocity of the superfluid component, v_s . In the following we describe a formalism suitable for the numerical solution of the transport balances in superfluid helium in three space dimensions.

2. Physical description – the system of PDE

We take as a starting point the system of partial differential equations discussed in [3], and refer to the same symbols used there. For the discussion of the numerical method, it is convenient to write the system in the following form:

$$\overline{\overline{\mathbf{m}}}\frac{\partial \mathbf{u}}{\partial t} + \overline{\overline{\mathbf{a}}}\nabla \cdot \mathbf{u} + \nabla \cdot \left(\overline{\overline{\mathbf{g}}}\nabla \cdot \mathbf{u}\right) + \overline{\overline{\mathbf{s}}}\mathbf{u} = \mathbf{q}$$
(1)

This is formally identical to a parabolic-hyperbolic differential equation in 3-D space, for which there is an established theory. The vector \mathbf{u} contains the unknown of the problem, i.e. the pressure and temperature, and the 3-D vectors of normal and superfluid velocity:

$$\mathbf{u} = \begin{bmatrix} p \\ \mathbf{v}_n \\ \mathbf{v}_s \\ T \end{bmatrix}$$
(2)

The matrix $\overline{\mathbf{m}}$ contains the coefficient of the time derivatives:

$$\overline{\overline{\mathbf{m}}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho_n & 0 & 0 \\ 0 & 0 & \rho_s & 0 \\ 0 & 0 & 0 & \rho C_y \end{bmatrix}$$
(3)

the matrix $\overline{\overline{a}}$ contains the coefficients of the convective fluxes (hyperbolic part), and is defined as follows:

$$\overline{\overline{\mathbf{a}}} = \begin{bmatrix} \frac{\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s}{\rho} & \rho_n c^2 + \phi T \rho_s s + \phi w^2 \frac{\rho_s \rho_n}{2\rho} & \rho_s c^2 - \phi T \rho_s s - \phi w^2 \frac{\rho_s \rho_n}{2\rho} & 0\\ \frac{\rho_n}{\rho} & \rho_n \mathbf{v}_n + \frac{\rho_s \rho_n}{\rho} \mathbf{w} & -\frac{\rho_s \rho_n}{\rho} \mathbf{w} & \rho_s s\\ \frac{\rho_s}{\rho} & -\frac{\rho_s \rho_n}{\rho} \mathbf{w} & \rho_s \mathbf{v}_s + \frac{\rho_s \rho_n}{\rho} \mathbf{w} & -\rho_s s\\ 0 & \rho_n \phi C_v T + T \rho_s s + w^2 \frac{\rho_s \rho_n}{2\rho} & \rho_s \phi C_v T - T \rho_s s - w^2 \frac{\rho_s \rho_n}{2\rho} & (\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s) C_v \end{bmatrix}$$
(4)

the matrix $\overline{\overline{g}}$ contains the coefficients of the diffusive fluxes (parabolic part), defined as:

 $\overline{\mathbf{s}}$ contains the coefficients of the non-linear sources, defined as:

$$\overline{\mathbf{s}} = \begin{bmatrix} 0 & -\phi A \rho_s \rho_n w^2 \mathbf{w} & \phi A \rho_s \rho_n w^2 \mathbf{w} & 0 \\ 0 & A \rho_s \rho_n w^2 & -A \rho_s \rho_n w^2 & 0 \\ 0 & -A \rho_s \rho_n w^2 & A \rho_s \rho_n w^2 & 0 \\ 0 & -A \rho_s \rho_n w^2 \mathbf{w} & A \rho_s \rho_n w^2 \mathbf{w} & 0 \end{bmatrix}$$
(6)

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and other forcing terms and explicit sources are included in the vector **q**:

$$\mathbf{q} = \begin{bmatrix} \phi q - \phi \overline{\overline{\tau}} \cdot \nabla \cdot \mathbf{v}_{n} - \rho c^{2} \mathbf{v}_{n} \nabla \frac{\rho_{n}}{\rho} - \rho c^{2} \mathbf{v}_{s} \nabla \frac{\rho_{s}}{\rho} - \phi T \mathbf{w} \nabla \rho_{s} s - \phi w^{2} \mathbf{w} \nabla \frac{\rho_{s} \rho_{n}}{2\rho} \\ -\nabla \cdot \overline{\overline{\tau}} + \rho_{n} \mathbf{g} \\ \rho_{s} \mathbf{g} \\ q - \overline{\overline{\tau}} \cdot \nabla \cdot \mathbf{v} - \rho \phi C_{v} T \mathbf{v}_{n} \nabla \frac{\rho_{n}}{\rho} - \rho \phi C_{v} T \mathbf{v}_{s} \nabla \frac{\rho_{s}}{\rho} - T \mathbf{w} \nabla \rho_{s} s - w^{2} \mathbf{w} \nabla \frac{\rho_{s} \rho_{n}}{2\rho} \end{bmatrix}$$
(7)

Note at this point that the drivers of mass and heat flux, i.e. pressure and temperature gradients, appear as a product of a quantity and the pressure or the temperature. This makes the mass and heat fluxes explicit in the formulation, which greatly stabilizes the numerical solution.

3. Formulation of the numerical solution

Several numerical techniques can be used to solve parabolic-hyperbolic systems of PDE, such as that of Eq. (1). We have chosen to use the Finite Element (FE) method in space [4], because it offers a very large flexibility in the modelling of complex 3-D geometries, and a generic time marching scheme of the type proposed by Beam and Warming [5].

Following FE formalism, we write weighted residuals of Eq. (1) over each element in the mesh. The first step is to approximate the variable \mathbf{u} using the nodal values, indicated as U:

$$\mathbf{u}(\mathbf{x}) \approx \mathbf{N}(\mathbf{x})\mathbf{U} \tag{8}$$

where the functions N(x) are the so-called shape functions of the element. For reasons of simplicity, we take the weight functions identical to the interpolation shape functions. The result of the weighted residual formulation is a system of Ordinary Differential Equations (ODE) in time:

$$\mathbf{M}\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{A} + \mathbf{G} + \mathbf{S})\mathbf{U} = \mathbf{Q}$$
(9)

where the matrices are the discretized version of the PDE system matrices, obtained by integration over the finite elements and assembly of the contributions of each element. Explicitly, the matrices are defined by the following over the 3-D domain of definition of the problem Ω :

$$M_{IJ} = \int_{\Omega J} N_I m_{IJ} N_J d\Omega \tag{10}$$

$$A_{IJ} = \int_{\Omega J} N_I a_{IJ} \nabla N_J d\Omega \tag{11}$$

$$G_{IJ} = \int_{\Omega J} \nabla N_I g_{IJ} \nabla N_J d\Omega$$
(12)

$$S_{IJ} = \int_{\Omega J} N_I s_{IJ} N_J d\Omega \tag{13}$$

$$Q_J = \int_{\Omega J} \nabla N_J q_J d\Omega \tag{14}$$

The system of ODE's Eq. (9) is solved with a multi-step algorithm of the Beam and Warming family [6]. The time discretization is performed as follows:

$$(1+\xi)\mathbf{M}\frac{\mathbf{U}^{n+1}-\mathbf{U}^n}{\Delta t^n}-\xi\mathbf{M}\frac{\mathbf{U}^n-\mathbf{U}^{n-1}}{\Delta t^{n-1}}+\theta\mathbf{H}(\mathbf{U}^{n+1}-\mathbf{U}^n)+\mathbf{H}\mathbf{U}^n+\phi\mathbf{H}(\mathbf{U}^n-\mathbf{U}^{n-1})=\mathbf{Q}$$
(15)

where we have introduced the matrix **H**, equal to the sum of the matrices **A**, **G** and **S**. The superscripts n-1, n and n+1 indicate variables at the corresponding time stations during integration, Δt^{n-1} and Δt^n are the time steps from time stations n-1 to n and from n to n+1 respectively. To linearize the solution of Eq. (15) all matrices and loads are evaluated from the known value **U**ⁿ and at a time t^* :

$$t^* = t^n + \theta \Delta t^n - \phi \Delta t^{n-1} \tag{16}$$

The parameters ξ , θ and ϕ are chosen so that the method is consistent and achieves the desired accuracy. Depending on the choice of the set of parameters, several known numerical schemes can be obtained [5].

4. Application to particle image velocimetry

As a first application of the numerical formulation, we aim at simulating the phenomena observed with the Particle Image Velocimetry (PIV) technique for an immersed particle in helium II [7]. Using the described numerical formulation, it will be possible to examine the details of the normal and superfluid helium flow, and compute the force on a particle immersed in a bath of helium II subjected to a heat flux. The main difficulty is that PDE under study is strongly non-linear due to the complexity of the interactions between the two types of fluids and the unique properties of the superfluid helium. We are presently examining the contribution of the single terms in the physical description, e.g. the nature and strength of the mutual friction, mutual influence of temperature and temperature gradients.

5. Conclusion

We have described the formalism for a 3-D solution of transport phenomena in superfluid helium. The Finite Element formulation contains all physical terms consistent with the mass, momentum and energy balance in the two-fluid approximation with mutual friction. The code implemented is presently under test, and we expect to provide results of relevance for the interpretation of experiments such as Particle Image Velocimetry (PIV) in simple configurations.

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